

Squariness in the special  $L$ -value  
or  
Squariness in the special  $L$ -value and special  
 $L$ -values of twists \*

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May 26, 2009

**Abstract**

Let  $N$  be a prime and let  $A$  be a quotient of  $J_0(N)$  over  $\mathbf{Q}$  associated to a newform such that the special  $L$ -value of  $A$  (at  $s = 1$ ) is non-zero. Suppose that the algebraic part of the special  $L$ -value of  $A$  is divisible by an odd prime  $q$  such that  $q$  does not divide the numerator of  $\frac{N-1}{12}$ . Then the Birch and Swinnerton-Dyer conjecture predicts that the  $q$ -adic valuations of the algebraic part of the special  $L$ -value of  $A$  and of the order of the Shafarevich-Tate group are both positive even numbers. Under a certain mod  $q$  non-vanishing hypothesis on special  $L$ -values of twists of  $A$ , we show that the  $q$ -adic valuations of the algebraic part of the special  $L$ -value of  $A$  and of the Birch and Swinnerton-Dyer conjectural order of the Shafarevich-Tate group of  $A$  are both positive even numbers. We also give a formula for the algebraic part of the special  $L$ -value of  $A$  over quadratic imaginary fields  $K$  in terms of the free abelian group on isomorphism classes of supersingular elliptic curves in characteristic  $N$  (equivalently, over conjugacy classes of maximal orders in the definite quaternion algebra over  $\mathbf{Q}$  ramified at  $N$  and  $\infty$ ) which shows that this algebraic part is a perfect square up to powers of the prime two and of primes dividing the discriminant of  $K$ . Finally, for an optimal elliptic curve  $E$ , we give a formula for the special  $L$ -value of the twist  $E_{-D}$  of  $E$  by a negative fundamental discriminant  $-D$ , which shows that this special  $L$ -value

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\*I would like to modify the title and include the words “and special  $L$ -values of twists”. I have included the earlier title in case it was used as an identifier for the paper.

<sup>†</sup>The author was partially supported by National Science Foundation Grant No. 0603668.

is an integer up to a power of 2, under some hypotheses. In view of the second part of the Birch and Swinnerton-Dyer conjecture, this leads us to the surprising conjecture that the square of the order of the torsion subgroup of  $E_{-D}$  divides the product of the order of the Shafarevich-Tate group of  $E_{-D}$  and the orders of the arithmetic component groups of  $E_{-D}$ , under certain mild hypotheses.

## 1 Introduction

Let  $A$  be an abelian variety over a number field  $F$ , and let  $L(A/F, s)$  denote the associated  $L$ -function, which we assume is defined over all of  $\mathbf{C}$  (this will be true in the cases we are interested in). Let  $\Omega(A/F)$  denote the quantity  $C_{A,\infty}$  in [Lan91, § III.5]; it is the “archimedean volume” of  $A$  over embeddings of  $F$  in  $\mathbf{R}$  and  $\mathbf{C}$  (e.g., if  $F = \mathbf{Q}$ , then it is the volume of  $A(\mathbf{R})$  computed using a generator for the highest exterior power of the group of invariant differentials on the Néron model of  $A$ ; the only other case we shall need is when  $F$  is a quadratic imaginary field, which is discussed at the beginning of Section 4). Let  $M_{\text{fin}}$  denote the set of finite places of  $F$ . Let  $\mathcal{A}$  denote the Néron model of  $A$  over the ring of integers of  $F$  and let  $\mathcal{A}^0$  denote the largest open subgroup scheme of  $\mathcal{A}$  in which all the fibers are connected. If  $v \in M_{\text{fin}}$ , then let  $\mathbf{F}_v$  denote the associated residue class field and let  $c_v(A/F) = [\mathcal{A}_{\mathbf{F}_v}(\mathbf{F}_v) : \mathcal{A}_{\mathbf{F}_v}^0(\mathbf{F}_v)]$ , the orders of the arithmetic component groups. Let  $\text{III}(A/F)$  denote the Shafarevich-Tate group of  $A$  over  $F$ . If  $F = \mathbf{Q}$ , then we will often drop the symbol “/ $F$ ” in the notation (thus  $\text{III}(A/\mathbf{Q})$  will be denoted  $\text{III}(A)$ , etc.). If  $B$  is an abelian variety over  $F$ , then we denote by  $B^\vee$  the dual abelian variety of  $B$ , and by  $B(F)_{\text{tor}}$  the torsion subgroup of  $B(F)$ . Suppose that  $L(A/F, 1) \neq 0$ . Then the second part of the Birch and Swinnerton-Dyer conjecture says the following (see [Lan91, § III.5]):

**Conjecture 1.1** (Birch and Swinnerton-Dyer).

$$\frac{L(A/F, 1)}{\Omega(A/F)} = \frac{|\text{III}(A/F)| \cdot \prod_{v \in M_{\text{fin}}} c_v(A/F)}{|A(F)_{\text{tor}}| \cdot |A^\vee(F)_{\text{tor}}|}. \quad (1)$$

We denote by  $|\text{III}(A/F)|_{\text{an}}$  the value of  $|\text{III}(A/F)|$  predicted by the conjecture above, and call it the *analytic order* of  $\text{III}(A/F)$ . Thus

$$|\text{III}(A/F)|_{\text{an}} = \frac{L(A/F, 1)}{\Omega(A/F)} \cdot \frac{|A(F)_{\text{tor}}| \cdot |A^\vee(F)_{\text{tor}}|}{\prod_{v \in M_{\text{fin}}} c_v(A/F)}.$$

We also call the ratio  $\frac{L(A/F,1)}{\Omega(A/F)}$ , the *algebraic part* of the special  $L$ -value of  $A_f$  over  $F$ ; in the contexts where we shall use this, it is known that the ratio is a rational number (and hence an algebraic number).

If  $N$  is a positive integer, then let  $X_0(N)$  denote the modular curve over  $\mathbf{Q}$  associated to  $\Gamma_0(N)$ , and let  $J_0(N)$  be its Jacobian. Let  $\mathbf{T}$  denote the subring of endomorphisms of  $J_0(N)$  generated by the Hecke operators (usually denoted  $T_\ell$  for  $\ell \nmid N$  and  $U_p$  for  $p \mid N$ ). If  $f$  is a newform of weight 2 on  $\Gamma_0(N)$ , then let  $I_f = \text{Ann}_{\mathbf{T}} f$  and let  $A_f$  denote the quotient abelian variety  $J_0(N)/I_f J_0(N)$  over  $\mathbf{Q}$ . We also denote by  $L(f, s)$  the  $L$ -function associated to  $f$  and by  $L(A_f, s)$  the  $L$ -function associated to  $A_f$ . It is known that  $\frac{L(A_f, 1)}{\Omega(A_f)}$  is a rational number.

Now fix a newform  $f$  of weight 2 on  $\Gamma_0(N)$  such that  $L(A_f, 1) \neq 0$ . Then by [KL89],  $A_f(\mathbf{Q})$  has rank zero and  $\text{III}(A_f)$  is finite. Thus the second part of the Birch and Swinnerton-Dyer conjecture becomes:

**Conjecture 1.2** (Birch and Swinnerton-Dyer).

$$\frac{L(A_f, 1)}{\Omega(A_f)} = \frac{|\text{III}(A_f)| \cdot \prod_{p \mid N} c_p(A_f)}{|A_f(\mathbf{Q})| \cdot |A_f^\vee(\mathbf{Q})|}, \quad (2)$$

Recall that an integer is said to be a *fundamental discriminant* if it is the discriminant of a quadratic field. The results of this paper concern the algebraic parts of the special  $L$ -values of  $A_f$  over  $\mathbf{Q}$ , of  $A_f$  over quadratic imaginary fields, and of twists of  $A_f$  by negative fundamental discriminants (over  $\mathbf{Q}$ ). In Section 2, when  $A_f$  is an elliptic curve, we give a formula for the special  $L$ -value of the twist of  $A_f$  by a negative fundamental discriminant, which shows that this special  $L$ -value is an integer, under some hypotheses. This leads us to the surprising conjecture that for such twists, the *square* of the order of the torsion subgroup divides the product of the order of the Shafarevich-Tate group and the orders of the arithmetic component groups, under certain mild hypotheses. In Section 3, under a certain mod  $q$  non-vanishing hypothesis on special  $L$ -values of twists of  $A_f$ , we show that when  $N$  is prime, for certain odd primes  $q$  that divide the algebraic part of the special  $L$ -value of  $A_f$  over  $\mathbf{Q}$ , the  $q$ -adic valuations of the algebraic part of the special  $L$ -value of  $A_f$  and of the Birch and Swinnerton-Dyer conjectural order of the Shafarevich-Tate group of  $A_f$  are both positive even numbers, in conformity with what the second part of the Birch and Swinnerton-Dyer conjecture predicts. In Section 4, for  $N$  prime, we give a formula for the algebraic part of the special  $L$ -value of  $A_f$  over quadratic imaginary fields  $K$  in terms of the free abelian group on isomorphism classes of supersingular

elliptic curves in characteristic  $N$  (equivalently over conjugacy classes of maximal orders in the definite quaternion algebra over  $\mathbf{Q}$  ramified at  $N$  and  $\infty$ ) which shows that this algebraic part is a perfect square away from the prime two and the primes dividing the discriminant of  $K$ . In Section 5, we give the proofs of two theorems mentioned in Sections 3 and 4. Finally, in Section 6, we give a formula for the determinant of the “complex period matrix” of an abelian variety, which is needed in the proof of the main theorem of Section 4. All the sections except Section 5 may be read independently of each other, although there is some cross referencing.

We now introduce some notation that will be used in various sections of this article. If  $\langle \cdot, \cdot \rangle : M \times M' \rightarrow \mathbf{C}$ , is a pairing between two  $\mathbf{Z}$ -modules  $M$  and  $M'$ , each of the same rank  $m$ , and  $\{\alpha_1, \dots, \alpha_m\}$  and  $\{\beta_1, \dots, \beta_m\}$  are bases of  $M$  and  $M'$  (respectively), then by  $\text{disc}(M \times M' \rightarrow \mathbf{C})$ , we mean  $\det(\langle \alpha_i, \beta_j \rangle)$ . Up to a sign,  $\text{disc}(M \times M' \rightarrow \mathbf{C})$  is independent of the choices of bases made in its definition, and in the rest of this paper,  $\text{disc}(M \times M' \rightarrow \mathbf{C})$  will be well defined only up to a sign (this ambiguity will not matter for our main results). We have a pairing

$$H_1(X_0(N), \mathbf{Z}) \otimes \mathbf{C} \times S_2(\Gamma_0(N), \mathbf{C}) \rightarrow \mathbf{C} \quad (3)$$

given by  $(\gamma, g) \mapsto \langle \gamma, g \rangle = \int_{\gamma} 2\pi i g(z) dz$  and extended  $\mathbf{C}$ -linearly. At various points in this article, we will consider pairings between two  $\mathbf{Z}$ -modules; unless otherwise stated, each such pairing is obtained in a natural way from (3). We have an involution induced by complex conjugation on  $H_1(A_f, \mathbf{Z})$ . Let  $H_1(A_f, \mathbf{Z})^+$  and  $H_1(A_f, \mathbf{Z})^-$  denote the subgroups of elements of  $H_1(A_f, \mathbf{Z})$  on which the involution acts as 1 and  $-1$  respectively. Let  $S_f = S_2(\Gamma_0(N), \mathbf{Z})[I_f]$ , let  $\Omega_{A_f}^+ = \text{disc}(H_1(A_f, \mathbf{Z})^+ \times S_f \rightarrow \mathbf{C})$ , and let  $\Omega_{A_f}^- = \text{disc}(H_1(A_f, \mathbf{Z})^- \times S_f \rightarrow \mathbf{C})$ . In each section below, we will continue to use the notation introduced in this section, unless mentioned otherwise.

## 2 Special $L$ -values of twists of elliptic curves

In this section, we give a formula for the special  $L$ -value of the twist of an optimal elliptic curve by a negative fundamental discriminant, which shows that this special  $L$ -value is an integer up to a power of 2, under certain hypotheses. This has some surprising implications from the point of view of the Birch and Swinnerton-Dyer conjecture, as we shall discuss.

We now recall some definitions for an elliptic curve  $A$  defined over  $\mathbf{Q}$ . If  $d$  is a square free integer, then  $A_d$  denotes the twist of  $A$  by  $d$ . Thus if  $y^2 = x^3 + ax + b$  with  $a, b \in \mathbf{Q}$  is a Weierstrass equation for  $A$ , then

$y^2 = x^3 + d^2ax + d^3b$  is a Weierstrass equation for  $A_d$ . If  $-D$  is a negative fundamental discriminant, we shall often consider the following hypothesis on  $(A, -D)$ :

(\*\*)  $-D$  is coprime to the discriminant of some Weierstrass equation  $y^2 = x^3 + Ax + B$  for  $E$  with  $A, B \in \mathbf{Z}$ .

Note that for every elliptic curve over  $\mathbf{Q}$ , there is a Weierstrass equation  $y^2 = x^3 + Ax + B$  with  $A, B \in \mathbf{Z}$ . If  $A$  is an elliptic curve over  $\mathbf{Q}$ , then let  $\omega_A$  denote an invariant differential on a global minimal Weierstrass model of  $A$ , which is unique upto sign. Now assume that  $A$  is an optimal elliptic curve, i.e., it is  $A_f$  for some newform  $f$  of weight 2 on  $\Gamma_0(N)$  for some  $N$ . Let  $\pi : X_0(N) \rightarrow A$  denote the associated parametrization. Then the space of pullbacks of differentials on  $A$  to  $X_0(N)$  is spanned by the differential  $2\pi i f(z)dz$ ; let  $\omega_f$  be the differential on  $A$  whose pullback is precisely  $2\pi i f(z)dz$ . Then  $\omega_A = c\omega_f$  for some rational number  $c_A$ , which is called the Manin constant of  $A$ .

**Lemma 2.1.** *Let  $E$  be an optimal elliptic curve over  $\mathbf{Q}$  and let  $-D$  be a negative fundamental discriminant such that  $(E, -D)$  satisfies hypothesis (\*\*). Then up to a sign,*

$$\Omega(E_{-D}) = c_E \cdot c_\infty(E_{-D}) \cdot \Omega_E^- / \sqrt{-D} ,$$

where  $c_\infty(E_{-D})$  is the number of connected components of  $E_{-D}(\mathbf{R})$ .

*Proof.* By hypothesis (\*\*), there is a Weierstrass equation  $y^2 = x^3 + Ax + B$  for  $E$  with  $A, B \in \mathbf{Z}$ , such that  $-D$  is coprime to the discriminant of this equation. Denote this equation by (a). If  $(x)$  denotes a Weierstrass equation for an elliptic curve, then we denote the associated discriminant by  $\Delta(x)$  and the associated invariant differential by  $\omega(x)$ . Replacing  $x$  by  $x/(\sqrt{-D})^2$  and  $y$  by  $y/(\sqrt{-D})^3$ , we get the Weierstrass equation  $y^2 = x^3 + D^2Ax - D^3B$  for  $E_{-D}$  (in fact, this transformation gives an isomorphism of  $E$  and  $E_{-D}$  over  $\mathbf{Q}(\sqrt{-D})$ ); denote this equation by (b). Then by [Sil92, Table III.1.2],  $\Delta(b) = D^6\Delta(a)$  and

$$\omega(b) = \omega(a)/(\sqrt{-D}). \quad (4)$$

Since  $D$  is squarefree and coprime to  $\Delta(a)$ , if  $p$  is a prime that divides  $D$ , then  $\text{ord}_p(\Delta(a)) = 0 < 12$ , and  $\text{ord}_p(\Delta(b)) = \text{ord}_p(D^6\Delta(a)) = 6 < 12$ . Thus by [Sil92, Rmk. VII.1.1], equations (a) and (b) are both minimal at the primes dividing  $D$ . Also, if  $p$  is a prime that does not divide  $D$ , then the coefficients of (a) and (b) have the same order at  $p$ . Thus, following the

proof of Prop. VIII.8.2 in [Sil92], there is a transformation  $x = u^2x' + r$ ,  $y = u^3y' + u^2sx' + t$  for some integers  $u, r, s$  and  $t$ , with  $u \neq 0$ , which converts both equations (a) and (b) to equations that are minimal at all primes. Denote these equations by (c) and (d) respectively; these are then global minimal Weierstrass equations for  $E$  and  $E_{-D}$  respectively. Hence  $\omega_E = \omega(c)$  and  $\omega_{E_{-D}} = \omega(d)$ . By [Sil92, p. 49],  $\omega(c) = u\omega(a)$  and  $\omega(d) = u\omega(b)$ . Using (4),  $\omega(d) = u\omega(b) = u\omega(a)/(\sqrt{-D}) = \omega(c)/(\sqrt{-D})$ .

Also, equation (b) was obtained from equation (a) by replacing  $x$  by  $x/(\sqrt{-D})^2$  and  $y$  by  $y/(\sqrt{-D})^3$ . Thus if  $(x, y)$  is a point on (b), then the corresponding point on (a) is given by  $(x/(\sqrt{-D})^2, y/(\sqrt{-D})^3)$ . Since the transformation used to go from (b) to (d) was the same as the one used to go from (a) to (c), we see that again, if  $(x, y)$  is a point on (d), then the corresponding point on (c) is given by  $(x/(\sqrt{-D})^2, y/(\sqrt{-D})^3)$ . Denote this map from points on (b) to points on (a) by  $T$  and let  $\sigma$  denote complex conjugation. Then if  $P = (x, y)$  is a point on (d) that is fixed by complex conjugation, i.e.  $\sigma(x, y) = (x, y)$ , then  $\sigma(T(P)) = \sigma(x/(\sqrt{-D})^2, y/(\sqrt{-D})^3) = (x/(\sqrt{-D})^2, -y/(\sqrt{-D})^3) = -T(P)$ . From this we see that if  $\gamma \in H_1(E_{-D}, \mathbf{Z})$  is a generator, then  $T(\gamma) \in H_1(E, \mathbf{Z})^-$ . It is easy to see that  $T$  is invertible, and so  $T(\gamma)$  is a generator of  $H_1(E, \mathbf{Z})^-$ .

Thus  $\Omega_E^- = \int_{T(\gamma)} \omega_f$  up to a sign, and using the change of variables given by the transformation  $T$ , we see that  $\int_\gamma \omega_{E_{-D}} = \int_{T(\gamma)} \omega_E/(\sqrt{-D})$ . Also, recall that  $\omega_E = c_E \omega_f$ . From the discussion above, we see that up to a sign,  $\Omega(E_{-D}) = c_\infty(E_{-D}) \cdot \int_\gamma \omega_{E_{-D}} = c_\infty(E_{-D}) \cdot \int_{T(\gamma)} \omega_E/(\sqrt{-D}) = c_\infty(E_{-D}) \cdot c_E \cdot \int_{T(\gamma)} \omega_f/(\sqrt{-D}) = c_\infty(E_{-D}) \cdot c_E \cdot \Omega_E^-/\sqrt{-D}$ , as was to be shown.  $\square$

Let  $N$  be a positive integer and let  $f$  be a newform of weight 2 on  $\Gamma_0(N)$ . Let  $-D$  be a negative fundamental discriminant that is coprime to  $N$  and let  $\epsilon_D = (\frac{-D}{\cdot})$  denote the quadratic character associated to  $-D$ . If  $f(q) = \sum_{n>0} a_n q^n$  is the Fourier expansion of  $f$ , then the twist of  $f$  by  $\epsilon_D$  is the modular form whose Fourier expansion is  $(f \otimes \epsilon_D)(q) = \sum_{n>0} \epsilon_D(n) a_n q^n$ . It is in fact a newform in  $S_2(ND^2, \epsilon_D^2)$  (considering that  $D$  is coprime to  $N$ ; see, e.g., p. 221 and p. 228 of [AL78] and the references in loc. cit.). Just as we associated an abelian variety  $A_f$  to  $f$ , one can associate to  $f \otimes \epsilon_D$  an abelian variety quotient  $A_{f \otimes \epsilon_D}$  of  $J_1(ND^2)$ , and moreover, if  $f_1, \dots, f_d$  are the Galois conjugates of  $f$ , then  $L(A_{f \otimes \epsilon_D}, 1) = \prod_i L(f_i \otimes \epsilon_D, 1)$  (see, e.g. p. 89 and p. 95 of [Roh97]).

**Proposition 2.2.** *Suppose  $f$  has integer Fourier coefficients, and let  $E$  denote the associated optimal elliptic curve quotient of  $J_0(N)$  over  $\mathbf{Q}$ . Suppose that  $(E, -D)$  satisfies hypothesis  $(**)$  mentioned at the beginning of this section. Then*

$$\frac{L(E_{-D}, 1)}{\Omega(E_{-D})} = \frac{L(A_{f \otimes \epsilon_D}, 1)}{c_E \cdot c_\infty(E_{-D}) \cdot \Omega_{A_f}^- / \sqrt{-D}} ,$$

where recall that  $c_\infty(E_{-D})$  is the number of connected components of  $E_{-D}(\mathbf{R})$ ,  $c_E$  is the Manin constant of  $E$ , and  $\Omega_{A_f}^-$  is as defined at the end of Section 1. In particular, if  $N$  is square free or if  $c_E = 1$  (as is conjectured), then

$$\frac{L(A_{f \otimes \epsilon_D}, 1)}{\Omega_{A_f}^- / \sqrt{-D}} = \frac{L(E_{-D}, 1)}{\Omega(E_{-D})} ,$$

up to a power of 2.

*Proof.* The first statement follows from Lemma 2.1 above, considering that  $L(E_{-D}, 1) = L(f \otimes \epsilon_D, 1) = L(A_{f \otimes \epsilon_D}, 1)$ . The second statement follows from the first, considering that  $c_\infty(E_{-D})$  is a power of 2, and if  $N$  is squarefree, then  $c_E$  is a power of 2 as well (by [Maz78, Cor. 4.1]).  $\square$

The modular symbol  $\sum_{b \bmod D} \epsilon_D(b) \{-\frac{b}{D}, \infty\}$  is an element of  $H_1(X_0(N), \mathbf{Z})^-$  by [Reb06, Lemma 5.2] (see also [Man71, §9.8–9.9]), and will be denoted by  $e_D$ . Let  $\pi$  denote the quotient map  $J_0(N) \rightarrow A_f$ , and let  $\pi_*$  denote the induced map  $H_1(J_0(N)(\mathbf{C}), \mathbf{Q}) \rightarrow H_1(A_f(\mathbf{C}), \mathbf{Q})$ . Let  $d = \dim A_f$ .

**Proposition 2.3.** *Assume that  $L(A_{f \otimes \epsilon_D}, 1) \neq 0$ . Then up to a power of 2,*

$$\frac{L(A_{f \otimes \epsilon_D}, 1)}{\Omega_{A_f}^- / (-D)^{d/2}} = |\pi_*(H_1(X_0(N), \mathbf{Z})^-) : \pi_*(\mathbf{T}e_D)| .$$

*Proof.* The proof is very similar to the proof of Theorem 2.1 in [Aga07]. The main thing to note is that if  $f_1, \dots, f_d$  are the Galois conjugates of  $f$ , then for  $i = 1, \dots, d$ , we have  $L(f_i \otimes \epsilon_D, 1) = \frac{\langle e_D, f_i \rangle}{\sqrt{-D}}$  (see, e.g., [Reb06, p. 254] or [Man71, Thm 9.9]), and so  $L(A_{f \otimes \epsilon_D}, 1) = \prod_i L(f_i \otimes \epsilon_D, 1) = \prod_i \frac{\langle e_D, f_i \rangle}{\sqrt{-D}}$ . Also, up to a power of 2,  $\pi_*(H_1(X_0(N), \mathbf{Z})^-) = H_1(A_f, \mathbf{Z})^-$ . Hence, up to a power of 2,

$$\begin{aligned} \frac{L(A_{f \otimes \epsilon_D}, 1)}{\Omega_{A_f}^- / (-D)^{d/2}} &= \frac{\prod_i \langle e_D, f_i \rangle}{\text{disc}(\pi_*(H_1(X_0(N), \mathbf{Z})^-) \times S_{f \rightarrow \mathbf{C}})} \\ &= \frac{\prod_i \langle e_D, f_i \rangle}{\text{disc}(\pi_*(\mathbf{T}e_D) \times S_{f \rightarrow \mathbf{C}})} \cdot |\pi_*(H_1(X_0(N), \mathbf{Z})^-) : \pi_*(\mathbf{T}e_D)| . \end{aligned}$$

One can see in a manner similar to the proof of formula (6) in the proof of Theorem 2.1 in [Aga07] that the first factor above is 1, as we explain next. The proposition then follows from the claim in the previous sentence.

There is a perfect pairing

$$\mathbf{T} \times S_2(\Gamma_0(N), \mathbf{Z}) \rightarrow \mathbf{Z}$$

which associates to  $(T, f)$  the first Fourier coefficient  $a_1(f|T)$  of the modular form  $f|T$  (see [Rib83, (2.2)]); this induces a pairing

$$\psi : \mathbf{T}/I_f \times S_f \rightarrow \mathbf{Z}, \quad (5)$$

which is also perfect (e.g., see [Aga07, Lemma 2.2]).

*Claim:* The map  $\mathbf{T} \rightarrow \mathbf{T}e_D$  given by  $t \mapsto te_D$  induces an isomorphism  $\mathbf{T}/I_f \xrightarrow{\cong} \mathbf{T}e_D/I_f e_D$ .

*Proof:* It is clear that the map  $\mathbf{T} \rightarrow \mathbf{T}e_D/I_f e_D$  given by  $t \mapsto te_D$  is surjective. All we have to show is that the kernel of this map is  $I_f$ . It is clear that the kernel contains  $I_f$ . Conversely, if  $t$  is in the kernel, then  $te_D \in I_f e_D$ ; let  $i \in I_f$  be such that  $te_D = ie_D$ . Then  $(t-i)e_D = 0$ , and thus  $\int_{(t-i)e_D} \omega_f = 0$ , i.e.,  $\int_{e_D} \omega_{(t-i)f} = 0$ . If the eigenvalue of  $f$  under  $(t-i)$  is  $\lambda$ , then this means  $\lambda \cdot L(f \otimes \epsilon_D, 1) = 0$ , i.e.,  $\lambda = 0$  (since  $L(f \otimes \epsilon_D, 1) \neq 0$ , considering that  $L(A_{f \otimes \epsilon_D}, 1) \neq 0$ ). Thus  $(t-i) \in I_f$ , i.e.,  $t \in I_f$ . This proves the claim.

We continue the proof of the theorem. In what follows,  $i, j, k$ , and  $\ell$  are indices running from 1 to  $d$ . Let  $\{g_k\}$  be a  $\mathbf{Z}$ -basis of  $S_f$  and let  $\{t_j\}$  be the corresponding dual basis of  $\mathbf{T}/I_f$  under the perfect pairing  $\psi$  in (5) above. Then by the claim above,  $\{t_j e_D\}$  is a  $\mathbf{Z}$ -basis for  $\mathbf{T}e_D/I_f e_D$ . Now  $g_k = \sum_i a_{ki} f_i$  for some  $\{a_{ki} \in \mathbf{C}\}$ . Let  $A$  be the matrix having  $(k, i)$ -th entry  $a_{ki}$ , and let  $(a^{-1})_{i\ell}$  denote the  $(i, \ell)$ -th element of the inverse of  $A$ . Then

$$\begin{aligned} & \text{disc}(\mathbf{T}e_D/I_f e_D \times S_f \rightarrow \mathbf{C}) \\ &= \det\{\langle t_j e_D, g_k \rangle\} = \det\{\langle e_D, g_k | t_j \rangle\} = \det\{\langle e_D, (\sum_i a_{ki} f_i) | t_j \rangle\} \\ &= \det\{\langle e_D, \sum_i a_{ki} a_1(f_i | t_j) f_i \rangle\} \quad (\text{since } f_i \text{'s are eigenvectors}) \\ &= \det\{\langle e_D, \sum_i a_{ki} \sum_\ell (a^{-1})_{i\ell} a_1(g_\ell | t_j) f_i \rangle\} \quad (\text{using } f_i = \sum_\ell (a^{-1})_{i\ell} g_\ell) \\ &= \det\{\langle e_D, \sum_i a_{ki} (a^{-1})_{ij} f_i \rangle\} \quad (\text{using } a_1(g_\ell | t_j) = \delta_{\ell j}) \\ &= \det\{\sum_i a_{ki} (a^{-1})_{ij} \langle e_D, f_i \rangle\} = \det\{\sum_i a_{ki} \langle e_D, f_i \rangle (a^{-1})_{ij}\} \\ &= \det(A \Delta A^{-1}) \quad (\text{where } \Delta = \text{diag}(\langle e_D, f_i \rangle)) \\ &= \det(\Delta) = \prod_i \langle e_D, f_i \rangle. \end{aligned}$$

This shows what we wanted and finishes the proof of the proposition.  $\square$



**Corollary 2.4.** *Let  $E$  be an optimal elliptic curve over  $\mathbf{Q}$  and let  $-D$  be a negative fundamental discriminant such that  $(E, -D)$  satisfies hypothesis  $(**)$  mentioned at the beginning of this section. Assume either that the Manin constant of  $E$  is one (as conjectured) or that  $N$  is squarefree. Then*

$$\frac{L(E_{-D}, 1)}{\Omega(E_{-D})} \in \mathbf{Z}[1/2].$$

*Proof.* This follows from Propositions 2.2 and 2.3.  $\square$

In view of the Birch and Swinnerton-Dyer conjecture (Conjecture 1.2 above) and the conjecture that the Manin constant is one, the corollary above suggests the following conjecture:

**Conjecture 2.5.** *Let  $E$  be an optimal elliptic curve over  $\mathbf{Q}$  of conductor  $N$  and let  $-D$  be a negative fundamental discriminant such that  $(E, -D)$  satisfies hypothesis  $(**)$  mentioned at the beginning of this section. Recall that  $E_{-D}$  denotes the twist of  $E$  by  $-D$ . Suppose  $L(E_{-D}, 1) \neq 0$ . Then  $|E_{-D}(\mathbf{Q})|^2$  divides  $|\text{III}(E_{-D})| \cdot \prod_{p|N} c_p(E_{-D})$ , up to a power of 2.*

Using the mathematical software sage, with its inbuilt Cremona's database for all elliptic curves of conductor up to 130000, we verified the conjecture above for all triples  $(N, E, D)$  such that  $N$  and  $D$  are positive integers with  $ND^2 < 130000$ , and  $E$  is an optimal elliptic curve of conductor  $N$ . In fact, we found that even if replace the hypothesis  $(**)$  with the potentially weaker hypothesis that  $\gcd(N, D) = 1$ , the conclusion of the conjecture above was true in all examples, even at the prime 2 (i.e., not just up to a power of 2). We also found that in all these examples, the odd part of  $|E_{-D}(\mathbf{Q})|^2$  divides  $\prod_{p|N} c_p(E_{-D})$ , and that if  $-D \neq -3$ , then  $|E_{-D}(\mathbf{Q})|$  is a power of 2. Table 1 below shows some interesting examples. The example of  $E = 105a1$  shows that  $|E_{-D}(\mathbf{Q})|^2$  does not divide  $\prod_{p|N} c_p(E_{-D})$  in general (but it does divide  $|\text{III}(E_{-D})| \cdot \prod_{p|N} c_p(E_{-D})$ ). Also, if  $-D = -3$ , it is not true that  $|E_{-D}(\mathbf{Q})|$  is a power of 2, as the example of  $E = 14a1$  shows. If we relax the assumption that  $\gcd(N, D) = 1$ , then it is no longer true that  $|E_{-D}(\mathbf{Q})|^2$  divides  $|\text{III}(E_{-D})| \cdot \prod_{p|N} c_p(E_{-D})$ , as the examples  $E = 21a1$  and  $E = 27a1$  show.

### 3 Special L-values over $\mathbf{Q}$

We assume in this section that  $N$  is prime. Let  $f$  be a newform of weight 2 on  $\Gamma_0(N)$ , and as before let  $A_f$  denote the associated newform quotient of  $J_0(N)$  over  $\mathbf{Q}$ . Let  $q$  be an odd prime that does not divide the numerator

Table 1:

$E$	$-D$	$ E_{-D}(\mathbf{Q})_{\text{tor}} $	$\prod_p c_p(E_{-D})$	$ \text{III}(E_{-D})_{\text{an}} $
14a1	-3	6	36	1
21a1	-7	4	8	1
27a1	-3	3	1	1
105a1	-11	2	2	4

of  $\frac{N-1}{12}$  but divides  $\frac{L(A_f,1)}{\Omega(A_f)}$ . Note that the denominator of  $\frac{L(A_f,1)}{\Omega(A_f)}$  divides the numerator of  $\frac{N-1}{12}$  (e.g., by [AS05, Prop. 4.6] and the fact that the order of the cusp  $(0) - (\infty) \in J_0(N)(\mathbf{C})$  is the numerator of  $\frac{N-1}{12}$  when  $N$  is prime), and so it makes sense to talk about whether  $q$  divides  $\frac{L(A_f,1)}{\Omega(A_f)}$  or not. In this section, we show that under a certain mod  $q$  non-vanishing hypothesis on special  $L$ -values of twists of  $A_f$ , the  $q$ -adic valuations of the algebraic part of the special  $L$ -value of  $A_f$  and of the Birch and Swinnerton-Dyer conjectural order of the Shafarevich-Tate group of  $A_f$  are both positive even numbers, in conformity with what the second part of the Birch and Swinnerton-Dyer conjecture predicts.

**Proposition 3.1.** *Let  $q$  be as above. Then  $q$  divides  $|\text{III}(A_f)|_{\text{an}}$ . If the Birch and Swinnerton-Dyer conjecture (2) is true, then  $\text{ord}_q\left(\frac{L(A_f,1)}{\Omega(A_f)}\right)$  and  $\text{ord}_q(|\text{III}(A_f)|)$  are both positive even numbers.*

*Proof.* By [Eme03, Theorem B] (and considering that the order of the cuspidal subgroup of  $J_0(N)$  is the numerator of  $\frac{N-1}{12}$  when  $N$  is prime),  $q$  does not divide  $\prod_{p|N} c_p(A_f)$  or  $|A_f(\mathbf{Q})| \cdot |A_f^\vee(\mathbf{Q})|$ . Thus if  $q$  divides  $\frac{L(A_f,1)}{\Omega(A_f)}$  then  $q$  divides  $|\text{III}(E)|_{\text{an}}$ . Now assume the Birch and Swinnerton-Dyer conjecture (2), so that  $q$  divides  $|\text{III}(E)|$ . As mentioned towards the end of §7.3 in [DSW03], if  $A_f^\vee[\mathfrak{q}]$  is irreducible for all maximal ideals  $\mathfrak{q}$  of  $\mathbf{T}$  with residue field of characteristic  $q$ , then the  $q$ -primary part of  $\text{III}(A_f^\vee)$  (and hence that of  $\text{III}(A_f)$ ) has order a perfect square. In our case, this irreducibility holds by [Maz77, Prop. 14.2], and thus  $\text{ord}_q(|\text{III}(A_f)|)$  is a positive even number. Moreover, as mentioned above,  $q$  does not divide any of the quantities other than  $|\text{III}(A_f)|$  on the right side of (2), and hence we see that  $\text{ord}_q\left(\frac{L(A_f,1)}{\Omega(A_f)}\right)$  is a positive even number.  $\square$

In particular, by Proposition 3.1 and its proof, if an odd prime  $q$  divides

$\frac{L(A_f, 1)}{\Omega(A_f)}$  or  $|\text{III}(A_f)|_{\text{an}}$ , but does not divide the numerator of  $\frac{N-1}{12}$ , then  $q^2$  (not just  $q$ ) is expected to divide  $\frac{L(A_f, 1)}{\Omega(A_f)}$  and  $|\text{III}(A_f)|_{\text{an}}$ .

Let  $-D$  be a negative fundamental discriminant, and as before, let  $\epsilon_D = \left(\frac{-D}{\cdot}\right)$  denote the associated quadratic character. Suppose that  $D$  is coprime to  $N$ . Then  $f \otimes \epsilon_D$  is a modular form of level  $ND^2$ . By a refinement of a theorem Waldspurger (see [LR97]), there exist infinitely many prime-to- $N$  discriminants  $-D$  such that  $L(A_{f \otimes \epsilon_D}, 1) \neq 0$ . Suppose  $D$  is such that  $L(A_{f \otimes \epsilon_D}, 1) \neq 0$ . By Proposition 2.3, the quantity  $\frac{L(A_{f \otimes \epsilon_D}, 1)}{\Omega_{A_f}^- / (-D)^{d/2}}$  is an integer up to a power of 2, so it makes sense to ask if an odd prime divides it. Also, if  $A_f$  is an elliptic curve and  $(A_f, -D)$  satisfy hypothesis (\*\*) mentioned at the beginning of Section 2, then by Proposition 2.2,  $\frac{L(A_{f \otimes \epsilon_D}, 1)}{\Omega_{A_f}^- / (-D)^{d/2}}$  is the algebraic part of the special  $L$ -value of the twist of  $A_f$  by  $-D$ , up to a power of 2.

**Theorem 3.2.** *Recall that the level  $N$  is assumed to be prime, and  $q$  is an odd prime which does not divide the numerator of  $\frac{N-1}{12}$ , but divides  $\frac{L(A_f, 1)}{\Omega(A_f)}$ . Assume that  $q$  satisfies the following hypothesis:*

(\*) *there exists a negative fundamental discriminant  $-D$  that is coprime to  $N$  such that  $L(A_{f \otimes \epsilon_D}, 1) \neq 0$  and  $q$  does not divide  $\frac{L(A_{f \otimes \epsilon_D}, 1)}{\Omega_{A_f}^- / (-D)^{d/2}}$ .*

*Then  $\text{ord}_q\left(\frac{L(A_f, 1)}{\Omega(A_f)}\right)$  and  $\text{ord}_q(|\text{III}(A_f)|_{\text{an}})$  are both positive and even.*

We shall prove Theorem 3.2 in Section 5. Assuming hypothesis (\*), in view of Proposition 3.1, Theorem 3.2 provides theoretical evidence towards the Birch and Swinnerton-Dyer conjectural formula (2). We will say more about the hypothesis (\*) later in this section.

**Proposition 3.3.** *Recall again that the level  $N$  is assumed to be prime. Suppose  $q$  is an odd prime that does not divide the numerator of  $\frac{N-1}{12}$  and there is a normalized eigenform  $g \in S_2(\Gamma_0(N), \mathbf{C})$  such that  $L(A_g, 1) = 0$  and  $f$  is congruent to  $g$  modulo a prime ideal over  $q$  in the ring of integers of the number field generated by the Fourier coefficients of  $f$  and  $g$ .*

(i) *If the first part of the Birch and Swinnerton-Dyer conjecture is true for  $A_g$ , then  $q^2$  divides  $|\text{III}(A_f)|$ .*

(ii) *Suppose  $q$  satisfies hypothesis (\*) of Theorem 3.2. Then  $q^2$  divides  $\frac{L(A_f, 1)}{\Omega(A_f)}$  and the Birch and Swinnerton-Dyer conjectural value of  $|\text{III}(A_f)|$ .*

*In particular  $\frac{L(A_f, 1)}{\Omega(A_f)} \equiv \frac{L(A_g, 1)}{\Omega(A_g)} \pmod{q^2}$ .*

*Proof.* If the first part of the Birch and Swinnerton-Dyer conjecture (on rank) is true for  $A_g$ , then considering that  $L(A_g, 1) = 0$ , we see that  $A_g$  has positive Mordell-Weil rank. Part (i) now follows from [Aga07, Thm 6.1]. By [Aga07, Prop. 1.5], the hypotheses of the proposition imply that  $q$  divides  $L(A_f, 1)/\Omega(A_f)$ . Thus part (ii) follows from the Theorem above.  $\square$

Subject to hypothesis (\*), the proposition above shows some consistency between the predictions of the two parts of the Birch and Swinnerton-Dyer conjecture. There is a general philosophy that congruences between eigenforms should lead to congruences between algebraic parts of the corresponding special  $L$ -values, and there are theorems in this direction (see [Wat99] and the references therein for more instances). However, these theorems prove congruences modulo primes, but not their powers. To our knowledge, part (ii) of Proposition 3.3 above is the first result of a form in which the algebraic parts of the special  $L$ -value are congruent modulo the *square* of a congruence prime.

In the rest of this section, we give some heuristic and computational evidence for why hypothesis (\*) might always hold when  $A_f$  is an elliptic curve, which we denote by  $E$ . Suppose that  $(E, -D)$  satisfies the hypothesis (\*\*) mentioned at the beginning of Section 2. Then, by Proposition 2.2,  $\frac{L(A_f \otimes \epsilon_D, 1)}{\Omega_{A_f}^- / (-D)^{d/2}}$  is the special  $L$ -value of the twisted elliptic curve  $E_{-D}$  up to a power of 2.

As mentioned before, by [Eme03, Theorem B],  $q$  does not divide the orders of the arithmetic component groups of  $E$ , and hence by [Pra08, Lem. 2.1],  $q$  does not divide the orders of the arithmetic component groups of  $E_{-D}$  either. Thus if one assumes the second part of the Birch and Swinnerton-Dyer conjecture for  $E_{-D}$ , then the only way  $q$  can divide  $\frac{L(A_f \otimes \epsilon_D, 1)}{\Omega_{A_f}^- / (-D)^{d/2}}$  is if  $q$  divides the order of  $\text{III}(E_{-D})$ .

Now there is no clear reason for  $q$  to divide the order of  $\text{III}(E_{-D})$  for *every*  $D$ . Kolyvagin has asked whether for a given elliptic curve  $A$  and a prime  $q$ , there is a twist of  $A$  such that  $q$  does not divide the order of the Shafarevich-Tate group of the twist (see Question A in [Pra08]). We are interested in the same question, but with the added restrictions that the level  $N$  is prime, the special  $L$ -value of the twist is nonzero, and that  $(E, -D)$  satisfies the hypothesis (\*\*).

We now report on what numerical data we could gather regarding this question. Since we do not know a general algorithm to compute the actual order of the Shafarevich-Tate group of an elliptic curve, we shall instead consider the analytic orders and assume the second part of the Birch and

Swinnerton-Dyer conjecture to pass from analytic orders of the Shafarevich-Tate groups to their actual orders.

Using the mathematical software sage, with its inbuilt Cremona's database for all elliptic curves of conductor up to 130000, we considered all tuples  $(N, E, p)$  such that  $N$  is an integer less than 130000,  $E$  is an elliptic curve of conductor  $N$  with  $|\text{III}(E)|_{\text{an}}$  divisible by an odd prime, and  $p$  is an odd prime that divides  $|\text{III}(E)|_{\text{an}}$ . For each such tuple, we looked for a negative fundamental discriminant  $-D$  such that  $L(E_{-D}, 1) \neq 0$ ,  $ND^2 < 130000$  (to stay within the range of Cremona's database), and  $D$  is coprime to the discriminant of a chosen Weierstrass equation  $y^2 = x^3 + Ax + B$  of  $E$  with  $A, B \in \mathbf{Z}$ . If we insisted on  $N$  being prime, then we found four tuples  $(N, E, p)$  as above; for two of them, we were able to find a  $D$  as above, in both of which  $p$  did not divide  $|\text{III}(E_{-D})|_{\text{an}}$ . If we allow  $N$  to be arbitrary, then we found 357 tuples  $(N, E, p)$  as above, and for 103 of them, we were able to find a  $D$  as above, among which in 101 cases,  $p$  did not divide  $|\text{III}(E_{-D})|_{\text{an}}$ . Of course, for the examples where we could not find a suitable  $D$  in the range of Cremona's tables, one may have to look beyond  $ND^2 = 130000$  to satisfy hypothesis (\*). Indeed, even for  $N$  as low as 681, which is the first level at which an elliptic curve has the analytic order of the Shafarevich-Tate group divisible by an odd prime, the number of negative fundamental discriminants  $-D$  such that  $\gcd(N, D) = 1$  and  $ND^2 < 130000$  is just 4. In any case, when we could find a  $D$  satisfying the requirements above, it was often the case that  $p$  did not divide  $|\text{III}(E_{-D})|_{\text{an}}$ . Thus the data above does encourage the belief that hypothesis (\*) always holds for elliptic curves (even for non-prime levels). For more general newform quotients  $A_f$ , we do not know how to do computations (but see the remark at the end of Section 4).

As mentioned above, we have to assume the second part of the Birch and Swinnerton-Dyer conjecture to pass from analytic orders of the Shafarevich-Tate groups to their actual orders. The careful reader would have noticed that we want to apply hypothesis (\*) to give evidence for the second part of the Birch and Swinnerton-Dyer conjecture, and at the same time we are assuming the conjecture to give some credence to the hypothesis. While this may sound like circular reasoning, the point is that the conjecture is being applied in different contexts, and also our reasoning is not intended in any way to be a part of a proof.

One would of course hope that hypothesis (\*) is proved independent of the second part of the Birch and Swinnerton-Dyer conjecture. While it is known that hypothesis (\*) does hold for all but finitely many primes  $q$  (e.g., see [OS98, Cor. 1]), it is not clear what that finite list of primes is. Also,

in [BO03, p.167-168], one finds a criterion for how big  $q$  needs to be, but the period they use (cf. [Bru99, §5]) differs from the period we use by an unknown algebraic number (cf. the discussion in [Koh85, Cor. 2], and [Pra08, Conj. 5.1]). Thus unfortunately the theoretical results mentioned in this paragraph do not help us much regarding hypothesis (\*).

## 4 Special L-values over quadratic imaginary fields

Let  $N$  be a positive integer. Let  $f$  be a newform of weight 2 on  $\Gamma_0(N)$ , and as before let  $A_f$  denote the associated newform quotient of  $J_0(N)$  over  $\mathbf{Q}$ . In this section, when  $N$  is prime, we give a formula for the algebraic part of the special  $L$ -value of  $A_f$  over quadratic imaginary fields  $K$  in terms of the free abelian group on isomorphism classes of supersingular elliptic curves in characteristic  $N$  (equivalently over conjugacy classes of maximal orders in the definite quaternion algebra over  $\mathbf{Q}$  ramified at  $N$  and  $\infty$ ) which shows that this algebraic part is a perfect square away from the prime two and the primes dividing the discriminant of  $K$ .

We start by recalling the definition of the “archemedian volume”  $\Omega(A_f/K)$  alluded to in the introduction. Let  $d = \dim A_f$  and let  $F$  be a number field. Let  $\omega_1, \dots, \omega_d$  be a basis of  $H^0(A_f, \Omega_{A_f/\mathbf{Q}}^1)$  associated to a  $\mathbf{Z}$ -basis of  $S_2(\Gamma_0(N), \mathbf{Z})[I_f]$ . Then  $\omega_1, \dots, \omega_d$  is also an  $F$ -basis of  $H^0(A_f, \Omega_{A_f/F}^1)$ . Let  $W$  denote the group of invariant differentials on the Néron model  $\mathcal{A}_{\mathcal{O}}$  of  $A_f$  over  $\mathcal{O}$ , the ring of integers of  $F$ . Then  $\wedge^d W = \mathfrak{c}(A_f/F) \cdot \wedge_i \omega_i$  for some fractional ideal  $\mathfrak{c}(A_f/F)$  of  $\mathcal{O}$  (cf. [Lan91, § III.5]). We will call the ideal  $\mathfrak{c}(A_f/F)$  the *Manin ideal* of  $A_f$  over  $F$ . If  $F = \mathbf{Q}$ , then the absolute value of a generator of the Manin ideal is just the Manin constant of  $A_f$  (as defined in [ARS06]) and is denoted  $c_{A_f}$ . If  $A_f$  is an elliptic curve, then this definition of the Manin constant agrees with the one given in Section 2 for optimal elliptic curves. The Manin constant  $c_{A_f}$  is conjectured to be one; it is known that  $c_{A_f}$  is an integer, and if  $p$  is a prime such that  $p^2 \nmid 2N$ , then  $p$  does not divide  $c_{A_f}$  (see [ARS06] for details).

**Lemma 4.1.** *The Manin ideal  $\mathfrak{c}(A_f/F)$  is supported on the set of maximal ideals  $\mathfrak{m}$  of  $\mathcal{O}$  such that the residue characteristic of  $\mathfrak{m}$  divides either  $c_{A_f}$  or the discriminant of  $\mathcal{O}$ .*

*Proof.* Suppose  $\mathfrak{m}$  is a maximal ideal of  $\mathcal{O}$  such that the residue characteristic  $\ell$  of  $\mathfrak{m}$  divides neither  $c_{A_f}$  nor the discriminant of  $\mathcal{O}$ . By [BLR90, §7.2, Cor. 2], over discrete valuation rings, the formation of Néron models is compatible with unramified extensions. Thus, considering that  $\ell$  is coprime

to the discriminant of  $\mathcal{O}$ ,  $H^0(\mathcal{A}_{\mathcal{O}}, \Omega_{\mathcal{A}_{\mathcal{O}}/\mathcal{O}}) \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{m}} = H^0(\mathcal{A}_{\mathcal{O}_{\mathfrak{m}}}, \Omega_{\mathcal{A}_{\mathcal{O}_{\mathfrak{m}}}/\mathcal{O}_{\mathfrak{m}}}) = H^0(\mathcal{A}_{\mathbf{Z}_{\ell}}, \Omega_{\mathcal{A}_{\mathbf{Z}_{\ell}}/\mathbf{Z}_{\ell}}) \otimes_{\mathbf{Z}_{\ell}} \mathcal{O}_{\mathfrak{m}} = H^0(\mathcal{A}_{\mathbf{Z}}, \Omega_{\mathcal{A}_{\mathbf{Z}}/\mathbf{Z}}) \otimes_{\mathbf{Z}} \mathbf{Z}_{\ell} \otimes_{\mathbf{Z}_{\ell}} \mathcal{O}_{\mathfrak{m}} = H^0(\mathcal{A}_{\mathbf{Z}}, \Omega_{\mathcal{A}_{\mathbf{Z}}/\mathbf{Z}}) \otimes_{\mathbf{Z}} \mathcal{O}_{\mathfrak{m}}$ . Thus, a  $\mathbf{Z}$ -basis  $\omega'_1, \dots, \omega'_d$  of  $H^0(\mathcal{A}_{\mathbf{Z}}, \Omega_{\mathcal{A}_{\mathbf{Z}}/\mathbf{Z}})$  is a  $\mathcal{O}_{\mathfrak{m}}$ -basis of  $H^0(\mathcal{A}_{\mathcal{O}}, \Omega_{\mathcal{A}_{\mathcal{O}}/\mathcal{O}}) \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{m}}$ . Since  $\ell$  does not divide  $c_{A_f}$ , we see that  $(\wedge^d \omega'_i) \otimes \mathbf{Z}_{\ell} = (\wedge^d \omega_i) \otimes \mathbf{Z}_{\ell}$ . Hence  $(\wedge^d \omega'_i) \otimes \mathcal{O}_{\mathfrak{m}} = (\wedge^d \omega_i) \otimes \mathcal{O}_{\mathfrak{m}}$ . In view of all this, it follows that  $\mathfrak{c}(A_f/F) \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{m}}$  is trivial, and the lemma follows.  $\square$

Let  $c_1, \dots, c_{2d}$  be a basis of  $H_1(A_f(\mathbf{C}), \mathbf{Z})$ . The *complex period matrix* of  $A_f$  (with respect to the chosen basis) is the  $2d \times 2d$  matrix  $A = (\int_{c_i} \omega_j, \int_{c_i} \bar{\omega}_j)$ . Recall that  $K$  is a quadratic imaginary field; let  $-D$  be its discriminant. The “archimedean volume” of  $A_f$  over  $K$  is

$$\Omega(A_f/K) = |\det(A)| \cdot N_{\mathbf{Q}}^K(\mathfrak{c}(A_f/F))/D^{d/2} \quad (6)$$

(this coincides with the definition of  $C_{A,\infty}$  in [Lan91, § III.5]).

Let  $N$  be prime in the rest of this section. We next give a formula for the ratio  $\frac{L(A_f/K, 1)}{\Omega(A_f/K)}$ , which is the left hand side of the Birch and Swinnerton-Dyer conjectural formula (Conjecture 1.1) for  $A_f$  over  $K$ .

Let  $\{E_0, E_1, \dots, E_g\}$  be a set of representatives for the isomorphism classes of supersingular elliptic curves in characteristic  $N$ , where  $g$  is the genus of  $X_0(N)$ . We denote the class of  $E_i$  by  $[E_i]$ . Let  $\mathcal{P}$  denote the divisor group supported on the  $[E_i]$  and let  $\mathcal{P}^0$  denote the subgroup of divisors of degree 0. For  $i = 1, 2, \dots, g$ , let  $R_i = \text{End } E_i$ . Each  $R_i$  is a maximal order in the definite quaternion algebra ramified at  $N$  and  $\infty$ , which we denote by  $\mathcal{B}$  and in fact, the  $R_i$ ’s are representatives of the conjugacy classes of maximal orders of  $\mathcal{B}$ . Moreover, setting  $I_i = \text{Hom}(E_0, E_i)$ , we see that the  $I_i$  are representatives for the isomorphism classes of right locally free rank one right modules over  $R_0$ . Let  $\mathcal{O}_{-D}$  denote the quadratic order of discriminant  $-D$ ,  $h(-D)$  the number of classes of  $\mathcal{O}_{-D}$ ,  $u(-D)$  the order of  $\mathcal{O}_{-D}^*/\langle \pm 1 \rangle$ , and  $h_i(-D)$  the number of optimal embeddings of  $\mathcal{O}_{-D}$  in  $R_i$  modulo conjugation by  $R_i^*$ . Following [Gro87], we define

$$\chi_D = \frac{1}{2u(-D)} \sum_{i=0}^g h_i(-D)[E_i] \in \mathcal{P} \otimes \mathbf{Q}.$$

Let  $w_i = |\text{Aut } E_i| = |R_i^*/\langle \pm 1 \rangle|$ . Define the Eisenstein element in  $\mathcal{P} \otimes \mathbf{Q}$  as  $a_E = \sum_{i=0}^g \frac{[E_i]}{w_i}$ . Let  $\chi_D^0 = \chi_D - \frac{12}{p-1} \deg(e_D) a_E$ . Let  $n = \text{numr}(\frac{N-1}{12})$ ; then  $n\chi_D^0 \in \mathcal{P}^0$ . Since the level  $N$  is prime, the Hecke algebra  $\mathbf{T}$  is semi-simple, and hence we have an isomorphism  $\mathbf{T} \otimes \mathbf{Q} \cong \mathbf{T}/I_f \otimes \mathbf{Q} \oplus B$  of  $\mathbf{T} \otimes \mathbf{Q}$ -modules for some  $\mathbf{T} \otimes \mathbf{Q}$ -module  $B$ . Let  $\pi$  denote element of  $\mathbf{T} \otimes \mathbf{Q}$  that is the projection on the first factor. We prove the following in Section 5:

**Theorem 4.2.** *Recall that the level  $N$  is prime. Let  $K$  be a quadratic imaginary field with discriminant  $-D$  that is coprime to  $N$ . If  $L(A_f/K, 1) \neq 0$ , then up to powers of primes dividing  $2D$ ,*

$$\frac{L(A_f/K, 1)}{\Omega(A_f/K)} = \frac{|\pi(\mathcal{P}^0) : \pi(\mathbf{T}n\chi_D^0)|^2}{N_{\mathbf{Q}}^K(\mathfrak{c}(A_f/K)) \cdot n^2}.$$

Moreover,  $\frac{L(A_f/K, 1)}{\Omega(A_f/K)}$  is a perfect square up to powers of primes dividing  $2D$ .

This addresses the issue raised in [Reb06, p. 236] that as of the writing of loc. cit., one did not have a way of expressing special  $L$ -values over  $K$  in terms of the module  $\mathcal{P}$ . Also, it may be possible to use the formula above for computations using Brandt matrices (cf. [Koh]). Note that up to powers of primes dividing  $2D$ ,  $\frac{L(A_f/K, 1)}{\Omega(A_f/K)}$  equals  $\frac{L(A_f, 1)}{\Omega(A_f)} \cdot \frac{L(A_f \otimes \epsilon_D, 1)}{\Omega_{A_f}^-/(-D)^{d/2}}$  (see formula (11) in Section 5). Thus if the formula in Theorem 4.2 could be used for computations, then considering that one already knows how to compute  $\frac{L(A_f, 1)}{\Omega(A_f)}$  (see [AS05, §4]), one could compute  $\frac{L(A_f \otimes \epsilon_D, 1)}{\Omega_{A_f}^-/(-D)^{d/2}}$  systematically and check whether the hypothesis (\*) of Theorem 3.2 holds in particular examples for odd primes  $q$  not dividing  $D$ .

## 5 Proofs of Theorems 3.2 and 4.2

In this section, we prove Theorems 3.2 and 4.2. We shall be using results from [Reb06], and details of some of the facts that we use here routinely may be found in loc. cit.

Let  $\mathcal{H}$  denote the complex upper half plane, and let  $\{0, i\infty\}$  denote the projection of the geodesic path from 0 to  $i\infty$  in  $\mathcal{H} \cup \mathbf{P}^1(\mathbf{Q})$  to  $X_0(N)(\mathbf{C})$ . We have an isomorphism

$$H_1(X_0(N), \mathbf{Z}) \otimes \mathbf{R} \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{C}}(H^0(X_0(N), \Omega^1), \mathbf{C}),$$

obtained by integrating differentials along cycles. Let  $e$  be the element of  $H_1(X_0(N), \mathbf{Z}) \otimes \mathbf{R}$  that corresponds to the map  $\omega \mapsto -\int_{\{0, i\infty\}} \omega$  under this isomorphism. It is called the *winding element*. By the Manin-Drinfeld Theorem, (see [Lan95, Chap. IV, Theorem 2.1] and [Man72]),  $e \in H_1(X_0(N), \mathbf{Z}) \otimes \mathbf{Q}$ . Also, since the complex conjugation involution on  $H_1(X_0(N), \mathbf{Z})$  is induced by the map  $z \mapsto -\bar{z}$  on the complex upper half plane, we see that  $e$  is invariant under complex conjugation. Thus  $e \in$



$H_1(X_0(N), \mathbf{Z})^+ \otimes \mathbf{Q}$ . Let  $H^+$  and  $H^-$  denote the subgroup of elements of  $H_1(X_0(N), \mathbf{Z})$  on which the complex conjugation involution acts as 1 and  $-1$  respectively.

Assume henceforth that  $N$  is prime (which is a hypothesis for the theorems that we want to prove). Consider the  $\mathbf{T}[1/2]$ -equivariant isomorphism

$$\Phi : \mathcal{P}^0[1/2] \otimes_{\mathbf{T}[1/2]} \mathcal{P}^0[1/2] \rightarrow H^+[1/2] \otimes_{\mathbf{T}[1/2]} H^-[1/2] \quad (7)$$

obtained from [Reb06, Prop. 4.6] (which says that both sides of (7) are isomorphic to  $S_2(\Gamma_0(N), \mathbf{Z})[1/2]$ , and whose proof relies on results of [Eme02]). By [Reb06, Thm 0.2], we have  $\Phi_{\mathbf{Q}}(\chi_D^0 \otimes_{\mathbf{T}_{\mathbf{Q}}} \chi_D^0) = e \otimes_{\mathbf{T}_{\mathbf{Q}}} e_D$ , where the subscript  $\mathbf{Q}$  stands for tensoring with  $\mathbf{Q}$  (this follows essentially from [Gro87, Cor 11.6], along with its generalization [Zha01, Thm 1.3.2]). Thus  $\Phi_{\mathbf{Q}}$  induces an isomorphism

$$\mathbf{T}[1/2](n\chi_D^0 \otimes_{\mathbf{T}[1/2]} n\chi_D^0) \cong \mathbf{T}[1/2]ne \otimes_{\mathbf{T}[1/2]} \mathbf{T}[1/2]ne_D. \quad (8)$$

Note that  $ne \in H^+$  by II.18.6 and II.9.7 of [Maz77].

Recall that since the level  $N$  is prime, the Hecke algebra  $\mathbf{T}$  is semi-simple, and hence we have an isomorphism  $\mathbf{T} \otimes \mathbf{Q} \cong \mathbf{T}/I_f \otimes \mathbf{Q} \oplus B$  of  $\mathbf{T} \otimes \mathbf{Q}$ -modules for some  $\mathbf{T} \otimes \mathbf{Q}$ -module  $B$ . Recall also that  $\pi$  denotes the element of  $\mathbf{T} \otimes \mathbf{Q}$  that is the projection on the first factor. In this section, if  $X$  and  $Y$  are  $\mathbf{T}$ -modules with  $Y \subseteq X$ , then we shall write  $|\pi(\frac{X}{Y})|$  for  $|\pi(X) : \pi(Y)|$ , which is an integer; we are doing this so that the formulas do not look too terrible.

**Proposition 5.1.**

$$\left| \pi \left( \frac{H^+[1/2]}{\mathbf{T}[1/2]ne} \right) \right| \cdot \left| \pi \left( \frac{H^-[1/2]}{\mathbf{T}[1/2]ne_D} \right) \right| = \left| \pi \left( \frac{H^+[1/2] \otimes_{\mathbf{T}} H^-[1/2]}{\mathbf{T}[1/2]ne \otimes_{\mathbf{T}} \mathbf{T}[1/2]ne_D} \right) \right|$$

*Proof.* By [Maz77, §15], if  $\mathfrak{m}$  is a Gorenstein maximal ideal of  $\mathbf{T}$  with odd residue characteristic, then  $H_{\mathfrak{m}}^+$  and  $H_{\mathfrak{m}}^-$  are free  $\mathbf{T}_{\mathfrak{m}}$ -modules of rank one. Since the level is prime, the only non-Gorenstein ideals are the ones lying over 2, a prime that we are systematically inverting anyway.

Let  $\mathfrak{m}$  be a maximal ideal of  $\mathbf{T}$  with odd residue characteristic. Let  $x$  be a generator of  $H_{\mathfrak{m}}^+$  as a free  $\mathbf{T}_{\mathfrak{m}}$ -module, and let  $y$  be a generator of  $H_{\mathfrak{m}}^-$  as a free  $\mathbf{T}_{\mathfrak{m}}$ -module. Then there exists  $t_1 \in \mathbf{T}_{\mathfrak{m}}$  such that  $ne = t_1x$  and  $t_2 \in \mathbf{T}_{\mathfrak{m}}$

such that  $ne_D = t_2y$ . We have

$$\begin{aligned}
\left| \frac{\pi(H_{\mathbf{m}}^+ \otimes_{\mathbf{T}_{\mathbf{m}}} H_{\mathbf{m}}^-)}{\pi(\mathbf{T}_{\mathbf{m}} ne \otimes_{\mathbf{T}_{\mathbf{m}}} \mathbf{T}_{\mathbf{m}} ne_D)} \right| &= \left| \frac{\pi(\mathbf{T}_{\mathbf{m}} x \otimes_{\mathbf{T}_{\mathbf{m}}} \mathbf{T}_{\mathbf{m}} y)}{\pi(\mathbf{T}_{\mathbf{m}} t_1 x \otimes_{\mathbf{T}_{\mathbf{m}}} \mathbf{T}_{\mathbf{m}} t_2 y)} \right| \\
&= \left| \frac{\pi(\mathbf{T}_{\mathbf{m}}(x \otimes_{\mathbf{T}_{\mathbf{m}}} y))}{t_1 t_2 \pi(\mathbf{T}_{\mathbf{m}}(x \otimes_{\mathbf{T}_{\mathbf{m}}} y))} \right| \\
&= \left| \frac{\pi(\mathbf{T}_{\mathbf{m}})}{t_2 t_1 \pi(\mathbf{T}_{\mathbf{m}})} \right| \\
&= \left| \frac{\pi(\mathbf{T}_{\mathbf{m}})}{t_1 \pi(\mathbf{T}_{\mathbf{m}})} \right| \cdot \left| \frac{\pi(t_1 \mathbf{T}_{\mathbf{m}})}{t_2 \pi(t_1 \mathbf{T}_{\mathbf{m}})} \right|.
\end{aligned}$$

*Claim:*

$$\left| \frac{\pi(t_1 \mathbf{T}_{\mathbf{m}})}{t_2 \pi(t_1 \mathbf{T}_{\mathbf{m}})} \right| = \left| \frac{\pi(\mathbf{T}_{\mathbf{m}})}{t_2 \pi(\mathbf{T}_{\mathbf{m}})} \right|.$$

*Proof.* Consider the map  $\psi : \pi(\mathbf{T}_{\mathbf{m}}) \rightarrow \pi(t_1 \mathbf{T}_{\mathbf{m}})/t_2 \pi(t_1 \mathbf{T}_{\mathbf{m}})$  given as follows: if  $t \in \mathbf{T}_{\mathbf{m}}$ , then  $\pi(t) \mapsto \pi(t_1 t)$ . If  $\pi(t)$  is in the kernel of  $\psi$ , then  $\pi(t_1 t) = \pi(t_2 t_1 t')$  for some  $t' \in \mathbf{T}_{\mathbf{m}}$ . Then  $\pi(t_1(t - t_2 t')) = 0$ , and since  $\pi(t_1) \neq 0$  (as  $L(A_f, 1) \neq 0$ ), we have  $\pi(t) = \pi(t_2 t')$ . Thus the kernel of  $\psi$  is  $t_2 \pi(\mathbf{T}_{\mathbf{m}})$ , which proves the lemma.  $\square$

Using the claim and the series of equalities above, we have

$$\begin{aligned}
\left| \frac{\pi(H_{\mathbf{m}}^+ \otimes_{\mathbf{T}_{\mathbf{m}}} H_{\mathbf{m}}^-)}{\pi(\mathbf{T}_{\mathbf{m}} ne \otimes_{\mathbf{T}_{\mathbf{m}}} \mathbf{T}_{\mathbf{m}} ne_D)} \right| &= \left| \frac{\pi(\mathbf{T}_{\mathbf{m}})}{t_1 \pi(\mathbf{T}_{\mathbf{m}})} \right| \cdot \left| \frac{\pi(\mathbf{T}_{\mathbf{m}})}{t_2 \pi(\mathbf{T}_{\mathbf{m}})} \right| \\
&= \left| \frac{\pi(\mathbf{T}_{\mathbf{m}} x)}{t_1 \pi(\mathbf{T}_{\mathbf{m}} x)} \right| \cdot \left| \frac{\pi(\mathbf{T}_{\mathbf{m}} y)}{t_2 \pi(\mathbf{T}_{\mathbf{m}} y)} \right| \\
&= \left| \frac{\pi(H_{\mathbf{m}}^+)}{\pi(\mathbf{T}_{\mathbf{m}} ne)} \right| \cdot \left| \frac{\pi(H_{\mathbf{m}}^-)}{\pi(\mathbf{T}_{\mathbf{m}} ne_D)} \right| \\
&= \left| \pi\left(\frac{H_{\mathbf{m}}^+}{\mathbf{T}_{\mathbf{m}} ne}\right) \right| \cdot \left| \pi\left(\frac{H_{\mathbf{m}}^-}{\mathbf{T}_{\mathbf{m}} ne_D}\right) \right|.
\end{aligned}$$

Since this is true for every  $\mathbf{m}$  with odd residue characteristic, we get the statement in the proposition.  $\square$

**Proposition 5.2.**

$$\left| \pi\left(\frac{\mathcal{P}^0[1/2] \otimes_{\mathbf{T}[1/2]} \mathcal{P}^0[1/2]}{\mathbf{T}[1/2](n\chi_D^0 \otimes_{\mathbf{T}[1/2]} n\chi_D^0)}\right) \right| = \left| \pi\left(\frac{\mathcal{P}^0[1/2]}{\mathbf{T}[1/2]n\chi_D^0}\right) \right|^2.$$

*Proof.* By [Eme02, Thm 0.5], if  $\mathfrak{m}$  is a Gorenstein maximal ideal of  $\mathbf{T}$ , then  $\mathcal{P}_{\mathfrak{m}}^0$  is a free  $\mathbf{T}_{\mathfrak{m}}$ -module of rank one; let  $x$  be a generator. Then  $n\chi_D^0 = tx$  for some  $t \in \mathbf{T}_{\mathfrak{m}}$ . Hence in a manner similar to the steps in the proof of Proposition 5.1, we have

$$\begin{aligned} \left| \pi \left( \frac{\mathcal{P}_{\mathfrak{m}}^0 \otimes_{\mathbf{T}[1/2]} \mathcal{P}_{\mathfrak{m}}^0}{\mathbf{T}_{\mathfrak{m}}(n\chi_D^0 \otimes_{\mathbf{T}[1/2]} n\chi_D^0)} \right) \right| &= \left| \pi \left( \frac{\mathbf{T}_{\mathfrak{m}}x \otimes_{\mathbf{T}[1/2]} \mathbf{T}_{\mathfrak{m}}x}{\mathbf{T}_{\mathfrak{m}}(tx \otimes_{\mathbf{T}[1/2]} tx)} \right) \right| = \left| \pi \left( \frac{\mathbf{T}_{\mathfrak{m}}}{t^2 \mathbf{T}_{\mathfrak{m}}} \right) \right| \\ &= \left| \pi \left( \frac{\mathbf{T}_{\mathfrak{m}}}{t \mathbf{T}_{\mathfrak{m}}} \right) \right|^2 = \left| \pi \left( \frac{\mathcal{P}_{\mathfrak{m}}^0}{\mathbf{T}_{\mathfrak{m}} n\chi_D^0} \right) \right|^2. \end{aligned}$$

Since this holds for every maximal ideal  $\mathfrak{m}$  of odd residue characteristic, we get the proposition.  $\square$

By formula (7), formula (8), Proposition 5.1, and Proposition 5.2, we have

$$\left| \pi \left( \frac{H^+[1/2]}{\mathbf{T}[1/2]ne} \right) \right| \cdot \left| \pi \left( \frac{H^-[1/2]}{\mathbf{T}[1/2]ne_D} \right) \right| = \left| \pi \left( \frac{\mathcal{P}^0[1/2]}{\mathbf{T}[1/2]n\chi_D^0} \right) \right|^2 \quad (9)$$

Let  $\Omega_{A_f}^+ = \text{disc}(H_1(A_f, \mathbf{Z})^+ \times S_f \rightarrow \mathbf{C})$ ; it differs from  $\Omega(A_f)$  by a power of 2 (by [Aga07, Lemma 2.4]). By the proof of Theorem 2.1 of [Aga07], we have

$$\left| \pi \left( \frac{H^+}{\mathbf{T}(ne)} \right) \right| = n \cdot \frac{L(A_f, 1)}{\Omega_{A_f}^+}.$$

Using this and Proposition 2.3, equation (9) says that up to a power of 2,

$$\frac{L(A_f, 1)}{\Omega_{A_f}^+} \cdot \frac{L(A_{f \otimes \epsilon_D}, 1)}{\Omega_{A_f}^- / (-D)^{d/2}} = \frac{1}{n^2} \cdot \left| \pi \left( \frac{\mathcal{P}^0[1/2]}{\mathbf{T}[1/2]n\chi_D^0} \right) \right|^2. \quad (10)$$

*Proof of Theorem 4.2.* We have  $L(A_f/K, s) = L(A_f, s) \cdot L(A_{f \otimes \epsilon_D}, s)$ , and by Corollary 6.2 in Section 6, we have  $\Omega(A_f/K) = N_{\mathbf{Q}}^K(\mathfrak{c}(A_f/K)) \cdot \Omega_{A_f}^+ \cdot \Omega_{A_f}^- / (-D)^{d/2}$ , up to a sign. Thus we have

$$\frac{L(A_f/K, 1)}{\Omega(A_f/K)} = \frac{1}{N_{\mathbf{Q}}^K(\mathfrak{c}(A_f/K))} \cdot \frac{L(A_f, 1)}{\Omega_{A_f}^+} \cdot \frac{L(A_{f \otimes \epsilon_D}, 1)}{\Omega_{A_f}^- / (-D)^{d/2} / (-D)^{d/2}}, \quad (11)$$

up to a sign. The first claim of Theorem 4.2 now follows from (10). The second claim follows from the first since  $N_{\mathbf{Q}}^K(\mathfrak{c}(A_f/K))$  is coprime to  $2D$  by Lemma 4.1, considering that by [Maz78, Cor. 4.1],  $c_{A_f}$  is a power of 2 since  $N$  is prime.  $\square$

*Proof of Theorem 3.2.* If an odd prime  $q$  divides  $\frac{L(A_f, 1)}{\Omega(A_f)}$  (which recall differs from  $\frac{L(A_f, 1)}{\Omega_{A_f}^+}$  by a power of 2) and  $q$  does not divide  $\frac{L(A_f \otimes \epsilon_D, 1)}{\Omega_{A_f}^- / (-D)^{d/2}}$ , then by (10),  $\text{ord}_q\left(\frac{L(A_f, 1)}{\Omega(A_f)}\right)$  is even (and positive). By [Eme03, Theorem B], we have  $|A_f(\mathbf{Q})| = |A_f^\vee(\mathbf{Q})|$  and this order divides the numerator of  $\frac{N-1}{12}$ . Thus if  $q$  does not divide the numerator of  $\frac{N-1}{12}$ , then from (2),  $\text{ord}_q(|\text{III}(A_f)|_{\text{an}})$  is positive and even. This proves Theorem 3.2.  $\square$

## 6 Appendix: period matrices

In this section, we give a formula for the determinant of the “complex period matrix” of an abelian variety. The result is probably well known, but we could not find a suitable reference.

Let  $Y$  be an abelian variety over  $\mathbf{Q}$  of dimension  $d$ . Let  $\omega_1, \dots, \omega_d$  be a basis of  $H^0(Y, \Omega_{Y/\mathbf{Q}}^1)$ . Let  $c_1, \dots, c_{2d}$  be a basis of  $H_1(Y(\mathbf{C}), \mathbf{Z})$ . We define the associated *complex period matrix* of  $Y$  as the  $2d \times 2d$  matrix  $A = (\int_{c_i} \omega_j, \int_{c_i} \overline{\omega}_j)$ ; this matrix depends on the choices of the bases made above.

We have an action of complex conjugation  $c$  on  $Y(\mathbf{C})$ , and hence on  $H_1(Y(\mathbf{C}), \mathbf{Z})$ . Let  $H_1(Y(\mathbf{C}), \mathbf{Z})^+$  denote the subgroup of elements of  $H_1(Y(\mathbf{C}), \mathbf{Z})$  that are fixed by complex conjugation, and let  $H_1(Y(\mathbf{C}), \mathbf{Z})^-$  denote the subgroup of elements  $x$  of  $H_1(Y(\mathbf{C}), \mathbf{Z})$  such that  $c(x) = -x$ . Let  $\gamma_1, \dots, \gamma_d$  be a basis of  $H_1(Y(\mathbf{C}), \mathbf{Z})^+$ , and let  $\gamma'_1, \dots, \gamma'_d$  be a basis of  $H_1(Y(\mathbf{C}), \mathbf{Z})^-$ . Let  $B$  denote the  $d \times d$  matrix whose  $(i, j)$ -th entry is  $\int_{\gamma_i} \omega_j$  and let  $C$  denote the  $d \times d$  matrix whose  $(i, j)$ -th entry is  $\int_{\gamma'_i} \omega_j$ .

**Lemma 6.1.** *We have  $\det A = \det B \cdot \det C$  up to a sign and up to a power of 2.*

*Proof.* Let  $A_{1,2}$  denote the  $d \times d$  matrix whose  $(i, j)$ -th entry is  $\int_{\gamma_i} \overline{\omega}_j$ , and let  $A_{2,2}$  denote the  $d \times d$  matrix whose  $(i, j)$ -th entry is  $\int_{\gamma'_i} \overline{\omega}_j$ . Consider the  $2d \times 2d$  matrix

$$A' = \begin{bmatrix} B & A_{1,2} \\ C & A_{2,2} \end{bmatrix}$$

Now the set  $\{\gamma_1, \dots, \gamma_d, \gamma'_1, \dots, \gamma'_d\}$  generates a subgroup of  $H_1(A(\mathbf{C}), \mathbf{Z})$  of index a power of 2, and thus up to a sign and up to a power of 2, we have

$$\det(A) = \det(A') . \tag{12}$$

Now if  $\gamma \in H_1(A(\mathbf{C}), \mathbf{Z})$ , and  $\bar{\gamma}$  denotes its complex conjugate, then for  $j = 1, \dots, d$ , since  $\omega_j$  is  $\mathbf{Q}$ -rational, we have  $\int_{\gamma} \bar{\omega}_j = \int_{\bar{\gamma}} \omega_j$ . In particular, if  $\gamma \in H_1(A(\mathbf{C}), \mathbf{Z})^+$ , then  $\int_{\gamma} \bar{\omega}_j = \int_{\gamma} \omega_j$ , and if  $\gamma \in H_1(A(\mathbf{C}), \mathbf{Z})^-$ , then  $\int_{\gamma} \bar{\omega}_j = -\int_{\gamma} \omega_j$ . Thus we see that  $A_{1,2} = B$  and  $A_{2,2} = -C$ . Thus

$$A' = \begin{bmatrix} B & B \\ C & -C \end{bmatrix}.$$

From this, we see that  $\det(A') = -2 \det(B) \det(C)$ . The lemma now follows from (12).  $\square$

We remark that the discussion above holds even if we replace  $\mathbf{Q}$  by  $\mathbf{R}$  throughout.

**Corollary 6.2.** *Let  $N$  be a positive integer. Let  $f$  be a newform of weight 2 on  $\Gamma_0(N)$ , and as before let  $A_f$  denote the associated newform quotient of  $J_0(N)$  over  $\mathbf{Q}$ . Recall that  $\Omega_{A_f}^+ = \text{disc}(H_1(A_f, \mathbf{Z})^+ \times S_f \rightarrow \mathbf{C})$ , and  $\Omega_{A_f}^- = \text{disc}(H_1(A_f, \mathbf{Z})^- \times S_f \rightarrow \mathbf{C})$ , where  $S_f = S_2(\Gamma_0(N), \mathbf{Z})[I_f]$ . Let  $K$  be a quadratic imaginary field of discriminant  $-D$ , and let  $\Omega(A_f/K)$  be the “complex period” of  $A_f$  over  $K$  as defined in formula (6) of Section 4. Then up to a sign,*

$$\Omega(A_f/K) = N_{\mathbf{Q}}^K(\mathfrak{c}(A_f/K)) \cdot \Omega_{A_f}^+ \cdot \Omega_{A_f}^- / (-D)^{d/2},$$

where  $\mathfrak{c}(A_f/K)$  is the Manin ideal of  $A_f$  over  $F$ , as defined at the beginning of Section 4.

*Proof.* Take  $Y = A_f$  in the discussion at the beginning of this section, and take  $\omega_1, \dots, \omega_d$  to be the differentials in  $H^0(A_f, \Omega_{A_f/\mathbf{Q}}^1)$  corresponding to a basis of  $S_f$ . Let the matrices  $A$ ,  $B$ , and  $C$  be as above, for the choices made in the previous sentence. Then by definition,  $\Omega(A_f/K) = |\det(A)| / (-D)^{d/2}$ ,  $\Omega_{A_f}^+ = \det(B)$ , and  $\Omega_{A_f}^- = \det(C)$ .

Now if  $\gamma \in H_1(A(\mathbf{C}), \mathbf{Z})$ , and  $\bar{\gamma}$  denotes its complex conjugate, then for  $j = 1, \dots, d$ , since  $\omega_j$  is  $\mathbf{Q}$ -rational, we have  $\overline{\int_{\gamma} \omega_j} = \int_{\bar{\gamma}} \omega_j$ . In particular, if  $\gamma \in H_1(A(\mathbf{C}), \mathbf{Z})^+$ , then  $\overline{\int_{\gamma} \omega_j} = \int_{\gamma} \omega_j$ , so  $\int_{\gamma} \omega_j$  is real. Hence all the entries of the matrix  $B$  are real. Hence  $|\det(B)| = \det(B)$  up to a sign. Similarly, if  $\gamma \in H_1(A(\mathbf{C}), \mathbf{Z})^-$ , then  $\overline{\int_{\gamma} \omega_j} = -\int_{\gamma} \omega_j$ , so  $\int_{\gamma} \omega_j$  is purely imaginary. Thus all the entries of the matrix  $C$  are purely imaginary. Hence  $|\det(C)| = (\sqrt{-1})^d \det(C)$  up to a sign.

Thus by Lemma 6.1 and the discussion in the two paragraphs above, we see that up to a sign,  $\Omega(A_f/K) = |\det(A)| / D^{d/2} = |\det(B)| \cdot |\det(C)| / D^{d/2} = \det(B) \cdot \det(C) \cdot (\sqrt{-1})^d / D^{d/2} = \det(B) \cdot \det(C) / (-D)^{d/2} = \Omega_{A_f}^+ \cdot \Omega_{A_f}^- / (-D)^{d/2}$ , as was to be shown.  $\square$

## Acknowledgments

This paper owes its existence to Loic Merel. He pointed out to us that Gross' formula [Gro87] should have some implications for the squareness of the order of the Shafarevich-Tate group, and also indicated the relevance of [Reb06]. The author's task was to work out the details and see what precise implications could be drawn. We would like to thank Loic Merel for suggesting this project as well as for several discussions regarding it. We are also grateful to J. Brown, W. Li, K. Ono, and K. Prasanna for some discussions. Some of the computations were done on `sage.math.washington.edu`, which is supported by National Science Foundation Grant No. DMS-0821725.

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